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THE HEWITT REALCOMPACTIFICATION OF PRODUCTS

BY

HARUTO OHTA

ABSTRACT. For a completely regular Hausdorff space X, υX denotes the Hewitt realcompactification of X. Given a topological property $\mathcal{P}$ of spaces, our interest is in characterizing the class $\mathfrak{R}(\mathcal{P})$ of all spaces X such that $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each $\mathcal{P}$-space Y. In the present paper, we obtain such characterizations in the case that $\mathcal{P}$ is locally compact and in the case that $\mathcal{P}$ is metrizable.

Introduction. All spaces considered in this paper are assumed to be completely regular Hausdorff and all maps are continuous. The Hewitt realcompactification υX of a space X is the unique realcompactification of X to which each real-valued continuous function on X admits a continuous extension. For details of Hewitt realcompactifications, the reader is referred to **[8]**. An important problem in the theory concerns when the relation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ is valid. Following **[23]**, **[30]**, we denote by $\mathfrak{R}$ (resp. $\mathfrak{R}(\mathcal{P})$) the class of all spaces X such that $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each space Y (resp. each $\mathcal{P}$-space Y), where $\mathcal{P}$ is a given property of spaces. It is known that: (Comfort **[4]**, **[5]**) a locally compact, realcompact space of nonmeasurable cardinal belongs to $\mathfrak{R}$; (Hušek **[12]**, **[14]** and McArthur **[23]**) every member of $\mathfrak{R}$ is realcompact; (Hušek **[13]**, **[14]**) every member of $\mathfrak{R}$ is of nonmeasurable cardinal; **[28]** every member of $\mathfrak{R}$ is locally compact. These facts characterize $\mathfrak{R}$ as precisely the class of locally compact, realcompact spaces of nonmeasurable cardinals. Further, in **[30]**, the author has tried to characterize $\mathfrak{R}(\mathcal{P})$ for various properties $\mathcal{P}$ of spaces, and has proved that $\mathfrak{R} = \mathfrak{R}$(metacompact) $= \mathfrak{R}$(subparacompact). It is the purpose of this paper to continue our study along this line, in particular, the following results are established:

(A) Both $\mathfrak{R}$(locally compact) and $\mathfrak{R}$(Moore) coincide with the class of all spaces of nonmeasurable cardinals whose Hewitt realcompactifications are locally compact.

(B) The class $\mathfrak{R}$(metrizable) consists precisely of all weak cb*-spaces, in the sense of Isiwata **[20]**, of nonmeasurable cardinals.

In §1, we present a technical theorem which is useful in finding a pair X, Y of spaces for which $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ fails. Our result (A) is then proved. We also give a positive answer to the following question of Hušek **[13**, p. 326**]**: Do there

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exist minimal cardinals $\mathfrak{m}$, $\mathfrak{n}$ for which $|X| = \mathfrak{m}$, $|Y| = \mathfrak{n}$ and $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y$. In §2, we prove the analogue for υX of the corollary to Glicksberg's theorem [**9**, Theorem 1]: For onto maps $f_i\colon X_i \to Y_i$ $(i = 1, 2)$, $\beta(Y_1 \times Y_2) = \beta Y_1 \times \beta Y_2$ holds whenever $\beta(X_1 \times X_2) = \beta X_1 \times \beta X_2$, where βX is the Stone-Čech compactification of X. It is shown that some additional conditions must be imposed in order that the analogous "υ" theorem holds. In §3, we apply our theory to prove (B), and also show that $\mathfrak{R}$(locally compact, metrizable) is precisely the class of all spaces of nonmeasurable cardinals. When studying the relation $\upsilon(X \times T) = \upsilon X \times \upsilon T$ with a metrizable factor T, the central issue is the weak cb* property in another factor X. It is proved that, in case X satisfies the countable chain condition and T is metrizable, the relation holds if and only if (i) either X or T is of nonmeasurable cardinal and (ii) either X is a weak cb*-space or T is locally compact. Finally a number of problems are posed in §4.

Throughout the paper, $\mathfrak{m}$ and $\mathfrak{n}$ denote cardinal numbers, and $\mathfrak{m}^+$ denotes the smallest cardinal greater than $\mathfrak{m}$. We let w, d, c and χ denote the following cardinal functions: weight, density, cellularity and character (cf. [**7**]). $|A|$ denotes the cardinality of a set A, and $\mathfrak{m}_1$ stands for the first measurable cardinal. Since $\mathfrak{m}_1$ (if it exists) is greater than any nonmeasurable cardinal, that $|A|$ is nonmeasurable is denoted by $|A| < \mathfrak{m}_1$. We also denote by $C(X)$ the set of real-valued continuous functions on a space X. For general terminology, see [**7**] and [**8**].

1. Characterizations of $\mathfrak{R}$(locally compact) and $\mathfrak{R}$(Moore). Two subsets A and B of a space X are said to be *completely separated in* X if there is $f \in C(X)$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. A family $\{F_\alpha\}$ of subsets of a space X is called *expandable* if there is a locally finite family $\{G_\alpha\}$ of open sets in X with $F_\alpha \subset G_\alpha$ for each α. We introduce a new class of expandable families.

1.1. Definition. A family $\{F_\alpha | \alpha \in A\}$ of subsets of a space X is $D(\mathfrak{m})$-*expandable* if there exists a locally finite family $\{G_\alpha | \alpha \in A\}$ of open sets in X with $F_\alpha \subset G_\alpha$ for each $\alpha \in A$ and each F_α is the union of at most $\mathfrak{m}$ subsets each of which is completely separated from $X - G_\alpha$.

If $\mathfrak{n} \geqslant \mathfrak{m}$, then a $D(\mathfrak{m})$-expandable family is $D(\mathfrak{n})$-expandable. As a space is completely regular Hausdorff, every expandable family in X is $D(|X|)$-expandable, and a uniformly locally finite family defined in [**17**] is $D(1)$-expandable (cf. [**27**]). Recall from [**17**] that a space X is *pseudo*-$\mathfrak{m}$-*compact* if each locally finite family of nonempty open sets in X has cardinality less than $\mathfrak{m}$. Pseudocompact spaces are known to be precisely pseudo-$\aleph_0$-compact spaces. The following theorem plays an essential role in our discussions.

1.2. Theorem. *Let $X \times Y$ be C-embedded in $X \times \upsilon Y$. If there exists a $D(\mathfrak{m})$-expandable family $\mathfrak{F}$ in Y, with $|\mathfrak{F}| = \mathfrak{n}$, such that $\bigcap \{\mathrm{cl}_{\upsilon Y} F | F \in \mathfrak{F}\} \neq \varnothing$, then each point $x \in X$, with $\chi(x, X) \leqslant \mathfrak{n}$, has a pseudo-$\mathfrak{m}$-compact neighborhood.*

Proof. Suppose on the contrary that there exists a point $x_0 \in X$, with $\chi(x_0, X) \leqslant \mathfrak{n}$, which has no pseudo-$\mathfrak{m}$-compact neighborhood. Let $\{G_\lambda | \lambda \in \Lambda\}$ be a neighborhood base at x_0 in X with $|\Lambda| = \mathfrak{n}$. Then, for each $\lambda \in \Lambda$, $\mathrm{cl}_X G_\lambda$ is not pseudo-$\mathfrak{m}$-compact, and thus there is a locally finite family $\{G'_{\lambda\mu} | \mu \in M_\lambda\}$ of nonempty open sets in $\mathrm{cl}_X G_\lambda$ with $|M_\lambda| = \mathfrak{m}$. Setting $G_{\lambda\mu} = G'_{\lambda\mu} \cap G_\lambda$ for each $\mu \in M_\lambda$, we have a locally finite family $\{G_{\lambda\mu} | \mu \in M_\lambda\}$ of nonempty open sets in X. It can be assumed without loss of generality that $x_0 \notin \bigcup \{G_{\lambda\mu} | \mu \in M_\lambda\}$. For each $\mu \in M_\lambda$, pick $x_{\lambda\mu} \in G_{\lambda\mu}$, and choose $f_{\lambda\mu} \in C(X)$ such that $f_{\lambda\mu}(x_{\lambda\mu}) = 0$ and $f_{\lambda\mu}(X - G_{\lambda\mu}) = \{1\}$. On the other hand, since $|\mathcal{F}| = \mathfrak{n}$, we may write $\mathcal{F} = \{F_\lambda | \lambda \in \Lambda\}$. Then there is a locally finite family $\{H_\lambda | \lambda \in \Lambda\}$ of open sets in Y with $F_\lambda \subset H_\lambda$ for each $\lambda \in \Lambda$. Each F_λ is a union of $\mathfrak{m}$ subsets each of which is completely separated from $Y - H_\lambda$, and so we express it by $F_\lambda = \bigcup \{F_{\lambda\mu} | \mu \in M_\lambda\}$, i.e., there is $g_{\lambda\mu} \in C(Y)$ such that $g_{\lambda\mu}(F_{\lambda\mu}) = \{0\}$ and $g_{\lambda\mu}(Y - H_\lambda) = \{1\}$. For each $\lambda \in \Lambda$ and each $\mu \in M_\lambda$, let us set

$$J_{\lambda\mu} = \{x_{\lambda\mu}\} \times F_{\lambda\mu}, \qquad K_{\lambda\mu} = G_{\lambda\mu} \times H_\lambda,$$
$$h_{\lambda\mu}((x, y)) = \min\{1, f_{\lambda\mu}(x) + g_{\lambda\mu}(y)\}, \qquad (x, y) \in X \times Y.$$

Then $h_{\lambda\mu} \in C(X \times Y)$, $h_{\lambda\mu}(J_{\lambda\mu}) = \{0\}$ and $h_{\lambda\mu}((X \times Y) - K_{\lambda\mu}) = \{1\}$. It is easily checked that $\mathcal{K} = \{K_{\lambda\mu} | \mu \in M_\lambda, \lambda \in \Lambda\}$ is locally finite in $X \times Y$. Therefore if we define a function h on $X \times Y$ by

$$h(p) = \inf\{h_{\lambda\mu}(p) | \mu \in M_\lambda, \lambda \in \Lambda\}, p \in X \times Y,$$

then h is continuous. Let us choose $y_0 \in \bigcap \{\mathrm{cl}_{\upsilon Y} F_\lambda | \lambda \in \Lambda\}$; then $y_0 \in \upsilon Y - Y$, because $\mathcal{F}$ is locally finite in Y. Now we show that h admits no continuous extension to the point $p_0 = (x_0, y_0) \in X \times \upsilon Y$. Let $U \times V$ be a given basic neighborhood of p_0 in $X \times \upsilon Y$. There is $\lambda \in \Lambda$ with $G_\lambda \subset U$, and $V \cap F_{\lambda\mu} \neq \varnothing$ for some $\mu \in M_\lambda$. Choose $y \in V \cap F_{\lambda\mu}$. Then both $p_1 = (x_{\lambda\mu}, y)$ and $p_2 = (x_0, y)$ belong to $U \times V$ and $h(p_1) = 0$, while $h(p_2) = 1$. This shows that h does not extend continuously to p_0, which contradicts the assumption that $X \times Y$ is C-embedded in $X \times \upsilon Y$. Hence the proof is complete.

1.3. Corollary. *Let $X \times Y$ be C-embedded in $X \times \upsilon Y$. If there exists a locally finite family $\mathcal{H}$ of nonempty open sets in Y, with $|\mathcal{H}| = \mathfrak{n}$, such that*

$$\bigcap \{\mathrm{cl}_{\upsilon Y} H | H \in \mathcal{H}\} \neq \varnothing,$$

then each point $x \in X$, with $\chi(x, X) \leqslant \mathfrak{n}$, has a pseudo-$c(Y)$-compact neighborhood.

Proof. Let $\mathcal{H} = \{H_\lambda | \lambda \in \Lambda\}$, and choose $y_0 \in \bigcap \{\mathrm{cl}_{\upsilon Y} H_\lambda | \lambda \in \Lambda\}$. For each $\lambda \in \Lambda$, by Zorn's lemma, there is a maximal disjoint family $\mathcal{F}_\lambda$ of nonempty open sets in H_λ such that each $F \in \mathcal{F}_\lambda$ is completely separated from $Y - H_\lambda$. Let us set $F_\lambda = \bigcup \{F | F \in \mathcal{F}_\lambda\}$. For each $\lambda \in \Lambda$, the maximality of $\mathcal{F}_\lambda$ implies that $y_0 \in \mathrm{cl}_{\upsilon Y} F_\lambda$. Since $|\mathcal{F}_\lambda| \leqslant c(Y)$, $\{F_\lambda | \lambda \in \Lambda\}$ is a $D(c(Y))$-expandable family in Y, with $|\Lambda| = \mathfrak{n}$, such that $\bigcap \{\mathrm{cl}_{\upsilon Y} F_\lambda | \lambda \in \Lambda\} \neq \varnothing$. Thus the corollary follows from Theorem 1.2.

1.4. Remark. Let us say that a family $\mathcal{G}$ of subsets of a space X *converges to* $x \in X$ if each neighborhood of x contains some member of $\mathcal{G}$, and that a subspace S of X is *relatively pseudo-$\mathfrak{m}$-compact in X* if each locally finite family $\mathcal{U}$ of nonempty open sets in X such that $S \cap U \neq \varnothing$ for each $U \in \mathcal{U}$ has cardinality less than $\mathfrak{m}$. The conclusion of Theorem 1.2 (resp. Corollary 1.3) can be generalized as follows: Each convergent family $\mathcal{G}$ of subsets of X, with $|\mathcal{G}| \leq \mathfrak{n}$, has a member which is relatively pseudo-$\mathfrak{m}$-compact (resp. relatively pseudo-$c(Y)$-compact) in X.

Our next work is to construct spaces Y which have a $D(\aleph_0)$-expandable family $\mathcal{F}$ such that $\bigcap\{\mathrm{cl}_{\upsilon Y}F | F \in \mathcal{F}\} \neq \varnothing$. A space is called *0-dimensional* if it has a base consisting of open-and-closed sets. For an ordinal α, we denote by $W(\alpha)$ the set of all ordinals less than α topologized with the order topology, and by ω_0 (resp. ω_1) the first infinite (resp. first uncountable) ordinal.

1.5. FACT. *For every infinite cardinal* $\mathfrak{n}$, *there exists a 0-dimensional locally compact space* $Y = Y_1(\mathfrak{n})$, *with* $|Y| = w(Y) = \mathfrak{n} \cdot \aleph_1$, *that has a $D(\aleph_0)$-expandable family* $\mathcal{F}$ *such that* $|\mathcal{F}| = \mathfrak{n}$ *and* $\bigcap\{\mathrm{cl}_{\upsilon Y}F | F \in \mathcal{F}\} \neq \varnothing$.

PROOF. Let $T_1 = W(\omega_1 + 1) \times W(\omega_0 + 1)$, and let $T_2 = \Lambda \cup \{\infty\}$ be the one point compactification of a discrete space Λ of cardinality $\mathfrak{n}$. We denote a base for the topology on T_i by $\mathcal{B}_i$ for $i = 1, 2$. Let $E = \{(\omega_1, \beta) | \beta \leq \omega_0\}$, and let Z' be the quotient space obtained from $R = T_1 \times T_2$ by collapsing the set $\{(\omega_1, \beta)\} \times T_2$ to a point $z(\beta) \in Z'$ for each $(\omega_1, \beta) \in E$. Let $\phi' \colon R \to Z'$ be the quotient map. Let Z_0 be the set Z', retopologized by letting $\bigcup\{\mathcal{B}(B) | B \in \mathcal{B}_1\}$ be a base, where

$$\mathcal{B}(B) = \begin{cases} \{\phi'(B \times T_2)\} & \text{if } B \cap E \neq \varnothing, \\ \{\phi'(B \times B') | B' \in \mathcal{B}_2\} & \text{if } B \cap E = \varnothing. \end{cases}$$

Then the natural map $\phi \colon R \to Z_0$ is continuous, and hence Z_0 is compact. Let us set

$$Z = Z_0 - \phi(\{((\gamma, \omega_0), \infty) | \gamma \leq \omega_1\}).$$

The space Z is a 0-dimensional locally compact space with $|Z| = w(Z) = \mathfrak{n} \cdot \aleph_1$. Since $z(\beta)$ is a P-point for each $\beta < \omega_0$, it is easily checked that Z is C-embedded in $Z \cup \{z_0\}$, where $z_0 = z(\omega_0)$, and so $z_0 \in \upsilon Z - Z$ by [**8**, 8.6]. Setting $D_\lambda = \phi(\{((\gamma, \omega_0), \lambda) | \gamma < \omega_1\})$ for each $\lambda \in \Lambda$, we have a discrete family $\{D_\lambda | \lambda \in \Lambda\}$ of closed subsets in Z such that $z_0 \in \bigcap \{\mathrm{cl}_{\upsilon Z} D_\lambda | \lambda \in \Lambda\} \neq \varnothing$. Define a subspace Y of the product space $Z \times W(\omega_0 + 1)$ by

$$Y = (Z \times \{\omega_0\}) \cup (\bigcup\{D_\lambda \times W(\omega_0 + 1) | \lambda \in \Lambda\}).$$

Then Y is a 0-dimensional locally compact space with $|Y| = w(Y) = \mathfrak{n} \cdot \aleph_1$, because Y is a closed subspace of $Z \times W(\omega_0 + 1)$. It remains to show the existence of a $D(\aleph_0)$-expandable family in Y satisfying the stated conditions. Since $Z \times \{\omega_0\}$ is C-embedded in Y, it follows from [**8**, 8.10(a)] that $\upsilon Z = \upsilon(Z \times \{\omega_0\}) \subset \upsilon Y$, and hence we may consider z_0 as an element of $\upsilon Y - Y$. Setting $F_\lambda = D_\lambda \times W(\omega_0)$ for each $\lambda \in \Lambda$, we have a discrete family $\mathcal{F} = \{F_\lambda | \lambda \in \Lambda\}$ of open sets in Y such that $z_0 \in \bigcap \{\mathrm{cl}_{\upsilon Y} F_\lambda | \lambda \in \Lambda\}$. Then, since each F_λ is a union of countably many open-and-closed subsets in Y, $\mathcal{F}$ is a $D(\aleph_0)$-expandable family in Y. Hence Y is proved to be the desired space.

1.6. FACT. *For every infinite cardinal* $\mathfrak{n}$, *there exists a 0-dimensional Moore space* $Y = Y_2(\mathfrak{n})$, *with* $|Y| = w(Y) = \mathfrak{n} \cdot \exp \aleph_0$, *that has a $D(\aleph_0)$-expandable family* $\mathcal{F}$ *such that* $|\mathcal{F}| = \mathfrak{n}$ *and* $\bigcap\{\mathrm{cl}_{\upsilon Y}F | F \in \mathcal{F}\} \neq \varnothing$.

PROOF. In [30], for every infinite cardinal $\mathfrak{n}$, we constructed a 0-dimensional Moore space $Z = Z(\mathfrak{n})$, with $|Z| = w(Z) = \mathfrak{n} \cdot \exp \aleph_0$, that has a discrete family $\mathfrak{D}$ of closed subsets such that $|\mathfrak{D}| = \mathfrak{n}$ and $\bigcap \{\mathrm{cl}_{\upsilon Z} D | D \in \mathfrak{D}\} \neq \varnothing$. The desired space $Y_2(\mathfrak{n})$ can be made from $Z(\mathfrak{n})$ by the same procedure as in the proof of 1.4.

For later use, we quote a theorem due to Hušek [13]:

1.7. THEOREM (HUŠEK). *Let Q be a discrete space. Then $\upsilon(P \times Q) = \upsilon P \times \upsilon Q$ holds if and only if either $|P| < \mathfrak{m}_1$ or $|Q| < \mathfrak{m}_1$ (i.e., either $|P|$ or $|Q|$ is nonmeasurable).*

We are now in a position to prove main theorems of this section. For the notion of locally pseudocompact spaces see [5]. We remark that the assumption $|X| < \mathfrak{m}_1$ of Theorem 1.8 is useful only for the implications (a) → (b) and (a) → (c).

1.8. THEOREM. *The following conditions on a space X with $|X| < \mathfrak{m}_1$ are equivalent:*

(a) *X is locally pseudocompact.*

(b) *$X \times Y$ is C-embedded in $X \times \upsilon Y$ for each 0-dimensional locally compact space Y with $w(Y) \leqslant \chi(X) \cdot \aleph_1$.*

(c) *$X \times Y$ is C-embedded in $X \times \upsilon Y$ for each 0-dimensional Moore space Y with $w(Y) \leqslant \chi(X) \cdot \exp \aleph_0$.*

PROOF. We proved in [28] that if X is a locally pseudocompact space of nonmeasurable cardinal, then $X \times Y$ is C-embedded in $X \times \upsilon Y$ for each k-space Y. Since both locally compact spaces and Moore spaces are k-spaces, (a) → (b) and (a) → (c) follow from this result. To prove (b) → (a) ((c) → (a)) suppose on the contrary that X is not locally pseudocompact at $x_0 \in X$. Let Y be the space $Y_1(\mathfrak{n})$ ($Y_2(\mathfrak{n})$) constructed in 1.5 (1.6), where $\mathfrak{n} = \chi(x_0, X)$. Then it follows from Theorem 1.2, that $X \times Y$ is not C-embedded in $X \times \upsilon Y$. This contradiction completes the proof.

1.9. THEOREM. *The following conditions on a space X are equivalent:*

(a) *υX is locally compact and $|X| < \mathfrak{m}_1$.*

(b) *$\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each 0-dimensional locally compact space Y with $w(Y) \leqslant \chi(\upsilon X) \cdot \aleph_1$.*

(c) *$\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each 0-dimensional Moore space Y with $w(Y) \leqslant \chi(\upsilon X) \cdot \exp \aleph_0$.*

PROOF. Since (a) → (b) and (a) → (c) follow from Hušek [14, Corollary (a), p. 177] (cf. also [28]), we prove only (b) → (a) and (c) → (a). By Theorem 1.8, υX is locally pseudocompact, and hence it is locally compact, because every pseudocompact realcompact space is compact [7, 3.11.1]. Suppose that $|X| \geqslant \mathfrak{m}_1$; then $\chi(\upsilon X) \geqslant \mathfrak{m}_1$ by [21, Theorem 2]. If we take for Y a discrete space of cardinality $\mathfrak{m}_1$, then it follows from Theorem 1.7 that $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y$. This contradicts (b) and (c) simultaneously. Hence the proof is complete.

In [14], Hušek proved that if X satisfies 1.9(a), then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each k-space Y. Therefore Theorem 1.9 tells us that both $\mathfrak{R}$(locally compact) and $\mathfrak{R}$(Moore) coincide with the class of spaces X such that υX is locally compact and $|X| < \mathfrak{m}_1$.

1.10. REMARKS. (1) Let Ψ be the space described in [8, 5I, p. 79]; Ψ is constructed as follows: Let $\mathfrak{E}$ be a maximal infinite almost-disjoint family of infinite subsets of the set N of integers. Then $|\mathfrak{E}| = \exp \aleph_0$. The space Ψ is the union of N with a new set $D = \{\omega_E | E \in \mathfrak{E}\}$ of distinct points endowed with the following topology: Each point of N is isolated, and a neighborhood of ω_E is any set containing ω_E and all but a finite number of points of E. It is well known that Ψ is a 0-dimensional pseudocompact (and hence $\beta\Psi = \upsilon\Psi$ by [8, 8A4, p. 125]) locally compact Moore space. In [25], Mrówka showed that $\mathfrak{E}$ can be chosen so that $\beta\Psi$ is the one point compactification. Then, dividing D into a disjoint family of countable infinite subsets, we have a discrete family $\mathfrak{D}$ of closed subsets in Ψ such that $|\mathfrak{D}| = \exp \aleph_0$ and $\bigcap \{\mathrm{cl}_{\upsilon\Psi} D' | D' \in \mathfrak{D}\} \neq \varnothing$. Thus, by the same method as in the proof of 1.5, we can make a 0-dimensional locally compact Moore space Y, with $w(Y) = \exp \aleph_0$, that has a $D(\aleph_0)$-expandable family $\mathfrak{F}$ such that $|\mathfrak{F}| = \exp \aleph_0$ and $\bigcap \{\mathrm{cl}_{\upsilon Y} F | F \in \mathfrak{F}\} \neq \varnothing$. This fact combined with Theorem 1.2 implies that the following condition (d) is also equivalent to 1.9(a) under the assumption that $\chi(\upsilon X) \leqslant \exp \aleph_0$.

(d) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ *holds for each* 0-*dimensional locally compact Moore space* Y *with* $w(Y) \leqslant \chi(\upsilon X) \cdot \exp \aleph_0$.

We do not know whether (d) implies 1.9(a) or not in general.

(2) We can apply our theory to answer the following question of Hušek [13, p. 326]: Do there exist spaces X, Y of cardinalities $\aleph_0$ and $\aleph_1$, respectively, such that $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y$? Let X be the space of rational numbers with the usual topology. We take for Y the space $Y_1(\aleph_0)$ constructed in 1.4. Then $|X| = w(X) = \aleph_0$ and $|Y| = w(Y) = \aleph_1$. Since X is not locally pseudocompact, it follows from Theorem 1.2 that $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y$.

2. Mapping theorems. In this section, we give mapping theorems which will be used in the next section. As is well known, for a map f: $X \to Y$, there exists a continuous extension υf: $\upsilon X \to \upsilon Y$ of f [8, 8.7]. If f_i: $X_i \to Y_i$ is a map for $i = 1, 2$, then the product map $f = f_1 \times f_2$ from $X_1 \times X_2$ to $Y_1 \times Y_2$ is defined by $f((x_1, x_2)) = (f_1(x_1), f_2(x_2))$ for $(x_1, x_2) \in X_1 \times X_2$.

2.1. THEOREM. *Let* f_i: $X_i \to Y_i$ $(i = 1, 2)$ *be onto maps. If* $\upsilon f_1 \times \upsilon f_2$ *is a quotient map from* $\upsilon X_1 \times \upsilon X_2$ *onto* $\upsilon Y_1 \times \upsilon Y_2$, *then* $\upsilon(X_1 \times X_2) = \upsilon X_1 \times \upsilon X_2$ *implies* $\upsilon(Y_1 \times Y_2) = \upsilon Y_1 \times \upsilon Y_2$.

More precisely, we have the following theorem:

2.2. THEOREM. *Let* F_i: $X_i^* \to Y_i^*$ $(i = 1, 2)$ *be onto maps such that* $F = F_1 \times F_2$ *is a quotient map, and let* X_i (*resp.* $Y_i = F_i(X_i)$) *be a dense C-embedded subspace of* X_i^* (*resp.* Y_i^*). *If* $X_1 \times X_2$ *is C-embedded in* $X_1^* \times X_2^*$, *then* $Y_1 \times Y_2$ *is C-embedded in* $Y_1^* \times Y_2^*$.

PROOF. Let us set $f_i = F_i | X_i$ $(i = 1, 2)$ and $f = f_1 \times f_2$. To show that $Y_1 \times Y_2$ is C-embedded in $Y_1^* \times Y_2^*$, let $g \in C(Y_1 \times Y_2)$. Since $h = g \circ f \in C(X_1 \times X_2)$, by our assumption, there exists $H \in C(X_1^* \times X_2^*)$ such that $H|(X_1 \times X_2) = h$. We shall show that $(*)$ H takes on the constant value t_p on $F^{-1}(p)$ for each $p \in Y_1^* \times Y_2^*$. Let $x \in X_1$; then $h(x, \cdot) = g(f_1(x), \cdot) \circ f_2$, where $h(x, \cdot) = h|(\{x\} \times X_2)$. Since $g(f_1(x), \cdot) \in C(Y_2)$, it has a continuous extension G_x over Y_2^*. Then, X_2 being dense in X_2^*, $H(x, \cdot) = G_x \circ F_2$. Hence it follows that $H(x, \cdot)$ is constant on $\{x\} \times F_2^{-1}(y)$ for each $y \in Y_2^*$. This implies that H is constant on $f_1^{-1}(y_1) \times F_2^{-1}(y_2)$ for each $(y_1, y_2) \in Y_1 \times Y_2^*$. Similarly, H is constant on $F_1^{-1}(y_1) \times f_2^{-1}(y_2)$ for each $(y_1, y_2) \in Y_1^* \times Y_2$. To see $(*)$, let $p = (y_1, y_2) \in Y_1^* \times Y_2^*$. Then it follows from these facts that

$$H(x, \cdot) = H(x', \cdot) \quad \text{for each } x, x' \in F_1^{-1}(y_1),$$
$$H(\cdot, x) = H(\cdot, x') \quad \text{for each } x, x' \in F_2^{-1}(y_2),$$

and from which $(*)$ is proved. Define a function G on $Y_1^* \times Y_2^*$ by $G(p) = t_p$ for each $p \in Y_1^* \times Y_2^*$. Then $H = G \circ F$ and $G|(Y_1 \times Y_2) = g$. Since F is a quotient map and H is continuous, it follows from [7, 2.4.2] that G is continuous. Hence our proof is complete.

Theorem 2.2 remains true if "C-embedded" is replaced by "C^*-embedded". Ishii proved in [**18**] that if $f\colon X \to Y$ is an open perfect onto map, then so is vf. This leads to the following corollary of Theorem 2.1.

2.3. COROLLARY. *If $f_i\colon X_i \to Y_i$ is an open perfect map onto Y_i for $i = 1, 2$, then $v(X_1 \times X_2) = vX_1 \times vX_2$ implies $v(Y_1 \times Y_2) = vY_1 \times vY_2$.*

The following theorem shows that in Theorem 2.1 the assumption that $vf_1 \times vf_2$ is quotient onto cannot be dropped, even when f_1 is an identity and f_2 is a perfect map. Recall from [**13**] that a space X is *pseudo-$\mathfrak{m}_1$-compact* if the cardinality of each locally finite family of nonempty open sets in X is nonmeasurable.

2.4. THEOREM. *Among the following conditions on a space X,* (a) $\to$ (b) $\to$ (c) *is valid. Conversely,* (c) $\to$ (a) *holds if $|X| < \mathfrak{m}_1$.*

(a) *vX is locally compact.*

(b) *For each space Y satisfying $v(X \times Y) = vX \times vY$ and each quotient image Z of Y, $v(X \times Z) = vX \times vZ$ holds.*

(c) *As in* (b), *with "perfect" instead of "quotient".*

PROOF. (a) $\to$ (b). Let Y be a space satisfying $v(X \times Y) = vX \times vY$, and let Z be the image of Y under a quotient map f. Since vX is locally compact, by Whitehead's theorem [7, 3.3.17], $\mathrm{id}_{vX} \times f$ is a quotient map, where id_{vX} is the identity map of vX. It follows from Theorem 2.2 that $X \times Z$ is C-embedded in $vX \times Z$. Hušek proved in [**13**] that if P is a locally compact, realcompact space, then $v(P \times Q) = vP \times vQ$ if and only if either $|P| < \mathfrak{m}_1$ or Q is pseudo-$\mathfrak{m}_1$-compact. If we apply this theorem to our case, then $|vX| < \mathfrak{m}_1$ or Y is pseudo-$\mathfrak{m}_1$-compact. If Y is pseudo-$\mathfrak{m}_1$-compact, so is Z. Hence it follows that $v(vX \times Z) = vX \times vZ$. Thus we have $v(X \times Z) = vX \times vZ$.

(b) $\to$ (c). Obvious.

(c) $\to$ (a). Suppose that $|X| < \mathfrak{m}_1$ and vX is not locally compact at $x_0 \in vX$.

Then, by [7, 3.11.1], x_0 has no pseudocompact neighborhood in υX. Let $\mathfrak{n} = \max\{|\upsilon X|, \chi(x_0, \upsilon X)\}$; then $\mathfrak{n} < \mathfrak{m}_1$. Let ω_α be the initial ordinal of $\mathfrak{n}^+$, and let $T = W(\omega_\alpha + 1) \times W(\omega_0 + 1)$. Let Λ be a discrete space of cardinality $\mathfrak{n}$, and let S_0 be the quotient space obtained from $R_0 = T \times \Lambda$ by collapsing the set $\{(\omega_\alpha, \beta)\} \times \Lambda$ to a point $s(\beta)$ for each $\beta \in E$, where $E = \{2n | n < \omega_0\} \cup \{\omega_0\}$. Let $g\colon R_0 \to S_0$ be the quotient map. Let us set $S = S_0 - \{s_0\}$, where $s_0 = s(\omega_0)$, and let $R = R_0 - g^{-1}(s_0)$. Then it is easily checked that $\upsilon S = S_0$ and $\upsilon R = R_0$. If we set

$$G = g(\{(\gamma, 2n) | \gamma \leqslant \omega_\alpha, n < \omega_0\} \times \Lambda),$$

then G is a cozero-set of S_0, and hence $G = G^* \cap S_0$ for some cozero-set G^* of βS_0 $(= \beta S)$. Let us set $Z = S \cup G^*$. We now need the following lemma:

2.5. LEMMA. *Let $X \supset X_1 \supset X_2$. Suppose that X_2 is dense in X and is C-embedded in X_1. Then, for each open set H of X, $X_2 \cup H$ is C-embedded in $X_1 \cup H$.*

The proof is left to the reader, since it requires only routine verification. We continue the proof of Theorem 2.4. By Lemma 2.5 and [7, 3.11.10], $\upsilon Z = S_0 \cup G^*$, and hence $s_0 \in \upsilon Z - Z$. Setting

$$F_\lambda = g(\{(\gamma, 2n + 1) | \gamma \leqslant \omega_\alpha, n < \omega_0\} \times \{\lambda\})$$

for each $\lambda \in \Lambda$, we obtain a locally finite family $\{F_\lambda | \lambda \in \Lambda\}$ of open sets in Z. Since each F_λ is a countable union of open-and-closed subsets of Z, $\{F_\lambda | \lambda \in \Lambda\}$ is a $D(\aleph_0)$-expandable family in Z such that $\bigcap\{\mathrm{cl}_{\upsilon Z} F_\lambda | \lambda \in \Lambda\} \ni s_0$. Since $\chi(x_0, \upsilon X) \leqslant |\Lambda|$, it follows from Theorem 1.2 that $\upsilon(X \times Z) \neq \upsilon X \times \upsilon Z$. For our end, it suffices to show that Z is the perfect image of a space Y satisfying $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$. There exists the extension $\beta g\colon \beta R_0 \to \beta S_0$ of g. Let us set $Y = R \cup H^*$, where $H^* = (\beta g)^{-1}(G^*)$, and set $f = (\beta g)|Y$. Then, since H^* is a cozero-set of βR_0 $(= \beta R)$, $\upsilon Y = R_0 \cup H^*$ by Lemma 2.5 and [7, 3.11.10]. Further it is easily checked that f is a perfect map from Y onto Z and Y is locally compact. Since $|Y| < \mathfrak{m}_1$, it follows from [5, Theorem 2.1] that $X \times Y$ is C-embedded in $\upsilon X \times Y$. It remains to show that $\upsilon X \times Y$ is C-embedded in $\upsilon X \times \upsilon Y$. Since $|\upsilon X| \leqslant \mathfrak{n}$, a similar argument to that of [8, 8.20] shows that $\upsilon X \times W(\omega_\alpha) \times W(\omega_0 + 1)$ is C-embedded in $\upsilon X \times W(\omega_\alpha + 1) \times W(\omega_0 + 1)$. Thus $\upsilon X \times R$ is C-embedded in $\upsilon X \times R_0$. Since $\upsilon X \times H^*$ is an open set of $\upsilon X \times \beta Y$, it follows from Lemma 2.5 that

$$(\upsilon X \times R) \cup (\upsilon X \times H^*) \quad (= \upsilon X \times Y)$$

is C-embedded in

$$(\upsilon X \times R_0) \cup (\upsilon X \times H^*) \quad (= \upsilon X \times \upsilon Y).$$

Hence the proof is complete.

2.6. REMARK. In case $|X| \geqslant \mathfrak{m}_1$, (c) $\to$ (a) of Theorem 2.4 need not be true. If D is a discrete space of cardinality $\mathfrak{m}_1$, then, by Theorem 1.6, D satisfies 2.4(c). But it is known [5, p. 115] that υD is not even a k-space.

The following corollary is proved by using Theorem 2.4 repeatedly.

2.7. COROLLARY. *Let $f_i\colon X_i \to Y_i$ $(i = 1, 2)$ be quotient onto maps. If both υX_1 and υY_2 are locally compact, then $\upsilon(X_1 \times X_2) = \upsilon X_1 \times \upsilon X_2$ implies $\upsilon(Y_1 \times Y_2) = \upsilon Y_1 \times \upsilon Y_2$.*

3. Characterizations of $\mathfrak{R}$(metrizable). A space X is called a *weak cb*-space* if for each decreasing sequence $\{F_n | n < \omega_0\}$ of regular closed sets in X with empty intersection, $\bigcap \{\mathrm{cl}_{vX} F_n | n < \omega_0\} = \varnothing$ holds, where a regular closed set is the closure of an open set. This notion was introduced by Isiwata **[20]** as a common generalization of realcompact spaces and weak cb-spaces in the sense of Mack and Johnson **[22]**. Since normal countably paracompact spaces, extremally disconnected spaces **[8**, 1H, p. 22] and pseudocompact spaces (or more generally, M'-spaces in the sense of Isiwata **[19]**) are weak cb-spaces, they are weak cb*-spaces. In this section, we prove the following theorem:

3.1. THEOREM. *The following conditions on a space X are equivalent*:
(a) *X is a weak cb*-space and* $|X| < \mathfrak{m}_1$.
(b) $v(X \times T) = vX \times vT$ *holds for each metrizable space T.*
(c) $v(X \times D(d(X))^\omega) = vX \times vD(d(X))^\omega$.

Here, $D(d(X))^\omega$ denotes the product of countably many copies of a discrete space of cardinality $d(X)$. Associated with each space X, there exist an extremally disconnected space $E(X)$ and a perfect irreducible map (i.e, a perfect map which takes proper closed subsets onto proper subsets) e_X from $E(X)$ onto X. The space $E(X)$ is unique up to homeomorphism and is called the *absolute* of X (cf. **[16]**, **[31]**). To prove Theorem 3.1, we make use of the following lemmas. The next lemma follows immediately from **[10**, Theorem 2.4] and **[11**, Proposition 1.2]; the first part also appears in **[15]**.

3.2. LEMMA. *A space X is a weak cb*-space if and only if $vE(X) = E(vX)$ holds. Moreover, in case $vE(X) = E(vX)$, then e_{vX} is the extension of e_X over $vE(X)$.*

3.3. LEMMA **[29]**. *Let X be a space and T a metrizable space. If either X is extremally disconnected or T is locally compact, then $X \times T$ is z-embedded in $\beta X \times T$ (i.e., each zero-set of $X \times T$ is the restriction to $X \times T$ of a zero-set of $\beta X \times T$).*

The next lemma is a corollary of Blair **[1**, Theorem 7.6]:

3.4. LEMMA (BLAIR). *Let $X \times Y$ be z-embedded in $\beta X \times Y$. If either $|X| < \mathfrak{m}_1$ or Y is pseudo-$\mathfrak{m}_1$-compact, then $v(X \times Y) = vX \times vY$ holds.*

PROOF OF THEOREM 3.1. (a) $\to$ (b). Let X be a weak cb*-space with $|X| < \mathfrak{m}_1$ and T a metrizable space. Since $|E(X)| < \mathfrak{m}_1$, it follows from Lemmas 3.3 and 3.4 that $v(E(X) \times T) = vE(X) \times vT$. By Lemma 3.2 ve_X $(= e_{vX})$ is a perfect map from $vE(X)$ onto vX, and so $ve_X \times \mathrm{id}_{vT}$ is perfect. Hence it follows from Theorem 2.1 that $v(X \times T) = vX \times vT$.

(b) $\to$ (c). Obvious.

(c) $\to$ (a). Suppose on the contrary that X is not a weak cb*-space. Then there is a locally finite sequence $\{G_n | n < \omega_0\}$ of open sets in X with $\bigcap \{\mathrm{cl}_{vX} G_n | n < \omega_0\} \neq \varnothing$. Since $c(X) \leqslant d(X)$, each point of $D(d(X))^\omega$ has no pseudo-$c(X)$-compact neighborhood, and $\chi(D(d(X))^\omega) = \aleph_0$. Hence it follows from Corollary 1.3 that $X \times D(d(X))^\omega$ is not C-embedded in $vX \times D(d(X))^\omega$. This contradicts (c). To prove that $|X| < \mathfrak{m}_1$, find a discrete family $\{G_\alpha | \alpha \in A\}$ of nonempty open sets in $D(d(X))^\omega$ with $|A| = d(X)$. Pick $t_\alpha \in G_\alpha$ for each $\alpha \in A$, and set $D = \{t_\alpha | \alpha \in A\}$. Then it is easily checked that $X \times D$ is C-embedded in $X \times D(d(X))^\omega$. Since $\upsilon D \subset \upsilon D(d(X))^\omega$, (c) implies $\upsilon(X \times D) = \upsilon X \times \upsilon D$. Hence it follows from Theorem 1.6 that $|X| < \mathfrak{m}_1$ or $|D|$ $(= d(X)) < \mathfrak{m}_1$. If $|D| < \mathfrak{m}_1$, then $|X| < \mathfrak{m}_1$ by [**8**, 12.5]. Thus the proof is complete.

3.5. REMARKS. (1) If $E(X) \times T$ is z-embedded in $\beta E(X) \times T$, then it follows from [**6**, Proposition 5.1] and [**2**, Corollary 3.6] that $E(X) \times T$ is C-embedded in $\upsilon E(X) \times T$. Therefore the proof of Theorem 3.1 shows that, more generally, a space X is a weak cb*-space if and only if $X \times T$ is C-embedded in $\upsilon X \times T$ for each metrizable space T.

(2) The product $X \times T$ of a weak cb*-space X with a metrizable space T need not be z-embedded in $\beta X \times T$. In fact, it was remarked in [**29**] that $\dim(X \times T) \leqslant \dim X + \dim T$ whenever $X \times T$ is z-embedded in $\beta X \times T$, while Wage showed in [**32**] that there exist a Lindelöf space (hence a weak cb*-space) X and a metrizable space T such that $\dim(X \times T) > \dim X + \dim T$.

(3) Lemmas 3.3 and 3.4 can be combined with Theorem 1.6 to yield the following result: *$\upsilon(X \times T) = \upsilon X \times \upsilon T$ holds for each locally compact, metrizable space T if and only if $|X| < \mathfrak{m}_1$.*

We conclude this section with a theorem, which gives conditions on X and Y necessary and sufficient that the relation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ be valid in a restrictive situation.

3.6. THEOREM. *Let X be a space satisfying the countable chain condition* (*i.e.*, *$c(X) \leqslant \aleph_0$*) *and T a metrizable space. Then $\upsilon(X \times T) = \upsilon X \times \upsilon T$ holds if and only if* (i) *either $|X| < \mathfrak{m}_1$ or $|T| < \mathfrak{m}_1$ and* (ii) *either X is weak cb* or T is locally compact.*

PROOF. *Necessity*: (i) is proved just like (c) $\to$ (a) in Theorem 3.1. If T is not locally compact, then T is not locally pseudocompact by [**7**, 3.10.21 and 4.1.17]. Thus it follows from Corollary 1.3 that X is a weak cb*-space.

Sufficiency: In case $|X| < \mathfrak{m}_1$, then the proof follows from Theorem 3.1 and Lemmas 3.3 and 3.4. In case $|T| < \mathfrak{m}_1$, then T is realcompact by [**8**, 15.20]. It follows from [**5**, Corollary 2.2] and 3.5(1) that $\upsilon(X \times T) = \upsilon X \times \upsilon T$. Hence the proof is complete.

3.7. REMARK. Theorem 3.6 fails to be valid if we drop the assumption $c(X) \leqslant \aleph_0$. To see this, we utilize the space Q of all rational numbers and the space Y_0 due to Comfort [**4**, p. 99]. The space Y_0 was constructed as the quotient space obtained from the product space

$$Z = W(\omega_0) \times W(\omega_1 + 1) \times W(\omega_1 + 1)$$

by identifying, for each $n < \omega_0$ and each $\gamma \leqslant \omega_1$, the two points (n, ω_1, γ) and $(n + 1, \gamma, \omega_1)$. Let $f\colon Z \to Y_0$ be the quotient map, and let us set $X = Y_0 - \{y_0\}$, where y_0 is the center point $f((0, \omega_1, \omega_1))$ $(= f((n, \omega_1, \omega_1)))$. Then he proved that $\upsilon X = Y_0$, and a similar argument assures us that $\upsilon(X \times Q) = \upsilon X \times Q$. Obviously Q is metrizable but not locally compact. It remains to show that X is not a weak cb*-space. Setting

$$F_n = f(\{i|i \geqslant n\} \times W(\omega_1 + 1) \times W(\omega_1 + 1)) \cap X$$

for each $n < \omega_0$, we obtain a decreasing sequence $\{F_n|n < \omega_0\}$ of regular closed sets in X with empty intersection. Then $y_0 \in \bigcap \{\mathrm{cl}_{\upsilon X} G_n|n < \omega_0\}$, and hence X is not a weak cb*-space.

4. Problems and remarks. Many interesting problems related to our results remain unsolved. Following **[20]**, we say that a space X is *υ-locally compact* if υX is locally compact.

4.1. Characterize $\mathfrak{R}(\upsilon$-locally compact). It is easy to see that

$$\mathfrak{R}(\upsilon\text{-locally compact}) = \mathfrak{R}(\text{pseudocompact}).$$

4.2. Characterize $\mathfrak{R}$(realcompact). Hušek **[12]**, **[14]** and McArthur **[23]** proved that each member X of this class, with $|X| < \mathfrak{m}_1$, is realcompact; however, the characterization is not yet known in complete form.

4.3. Characterize $\mathfrak{R}$(weak cb*). We note that it follows from Lemma 3.2 and Theorem 2.1 that $\mathfrak{R}$(weak cb*) $=$ $\mathfrak{R}$(extremally disconnected). Moreover, since the space Y constructed in the proof of **[23**, Theorem 5.2] is a weak cb*-space, every member of $\mathfrak{R}$(weak cb*) is realcompact.

4.4. Find conditions on X and T necessary and sufficient that $\upsilon(X \times T) = \upsilon X \times \upsilon T$ be valid in the case where T is a metrizable space.

4.5. Let f_i: $X_i \to Y_i$ $(i = 1, 2)$ be onto maps. When does $\upsilon(Y_1 \times Y_2) = \upsilon Y_1 \times \upsilon Y_2$ imply $\upsilon(X_1 \times X_2) = \upsilon X_1 \times \upsilon X_2$?

4.6. Remark. Let f: $Y \to Z$ be a perfect onto map. Then $\upsilon(X \times Z) = \upsilon X \times \upsilon Z$ does not necessarily imply $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$, even when $\mathrm{id}_{\upsilon X} \times \upsilon f$ is a quotient onto map and υX is compact. To see this, let us set $X = W(\omega_1)$; then by **[23**, Theorem 5.5] there exists a realcompact space Y_1 such that $\upsilon(X \times Y_1) \neq \upsilon X \times \upsilon Y_1$. By **[26**, Corollary 2.3], Y_1 can be embedded as a closed subspace of a pseudocompact space Y_2. Let i: $Y_1 \to Y_2$ be the embedding. Let us set $Y = Y_1 \oplus Y_2$ and $Z = Y_2$, where $\oplus$ means the topological sum. Define a map f: $Y \to Z$ by $f(y) = i(y)$ if $y \in Y_1$ and $f(y) = y$ if $y \in Y_2$. Then f is a perfect onto map and υf is a quotient map from υY $(= Y_1 \oplus \upsilon Y_2)$ onto υZ $(= \upsilon Y_2)$. Since X is locally compact, it follows from [7, 3.10.26] that $X \times Z$ is pseudocompact, and hence

$$\upsilon(X \times Z) = \upsilon X \times \upsilon Z$$

holds by Glicksberg's theorem **[9]**. On the other hand, $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y$ obviously. Further, υX being compact, it follows from [7, 3.3.17] that $\mathrm{id}_{\upsilon X} \times \upsilon f$ is a quotient onto map.

4.7. Find characterizations of an onto map f: $X \to Y$ for which υf: $\upsilon X \to \upsilon Y$ is an onto biquotient map in the sense of Michael **[24]**. We are interested in this problem in view of 4.8(3) below.

4.8. Remark. It seems that the classes $\mathfrak{R}(\mathcal{P})$ considered above have several common properties. Finally, we list some of these below. Each assertion follows from the results in the bracket.

(1) $\mathfrak{R}(\mathcal{P})$ includes all locally compact, realcompact spaces of nonmeasurable cardinals [**5**, Corollary 2.2].

(2) $\mathfrak{R}(\mathcal{P})$ is closed under cozero-subspaces [**3**, 3.2].

(3) $\mathfrak{R}(\mathcal{P})$ is closed under open perfect images (Corollary 2.3); more generally, if $vf\colon vX \to vY$ is an onto biquotient map, then $Y \in \mathfrak{R}(\mathcal{P})$ whenever $X \in \mathfrak{R}(\mathcal{P})$ ([**24**, Theorem 1.2] and Theorem 2.1).

(4) If each $\mathcal{P}$-space is v-locally compact, then $\mathfrak{R}(\mathcal{P})$ is closed under quotient images (Theorem 2.4).

(5) If $X \in \mathfrak{R}(\mathcal{P})$ and Y is a locally compact, realcompact space with $|Y| < \mathfrak{m}_1$, then $X \times Y \in \mathfrak{R}(\mathcal{P})$ [**5**, Corollary 2.2].

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, SAKURAMURA IBARAKI, 300-31 JAPAN

Current address: Faculty of Education, Shizuoka University, Ohya, Shizuoka, 422 Japan